

Structure of wrap groups of non-archimedean fibers.

S.V. Ludkovsky

17 November 2011 ^{*}

Abstract

This article is devoted to the investigation of structure of wrap groups of fiber bundles over ultra-normed infinite fields and more generally over Cayley-Dickson algebras. Iterated wrap groups are studied as well. Their smashed products are constructed and studied.

1 Introduction.

This article is a continuation of the previous one [15], where wrap groups of non-archimedean fiber bundles were defined and their existence was proved and some their properties were outlined. Below other structure theorems are formulated and proved.

Wrap groups of fiber bundles considered in this paper are constructed with the help of families of mappings from a fiber bundle with marked points into another fiber bundle with a marked point over an ultra-normed field \mathbf{K} or the octonion algebra \mathcal{A}_3 modeled on \mathbf{K} . This paper continues previous works on this theme, where generalized loop groups of manifolds over \mathbf{R} , \mathbf{C} and \mathbf{H} were investigated, but neither for fibers nor over octonions [9, 14, 12, 13].

Applications of quaternions in mathematics and physics can be found in [2, 5, 8].

^{*}Mathematics subject classification (1991 Revision) 22-99, 46S10, 54H15 and 57S20.

[†]keywords: wrap group; fiber bundle; infinite field; non-archimedean norm; Cayley-Dickson algebra

In this article wrap groups of different classes of smoothness are considered.

In particular, geometric loop groups have important applications in modern physical theories (see [7, 16] and references therein). Groups of loops are also intensively used in gauge theory. Wrap groups defined with the help of families of mappings from a manifold M into another manifold N with a dimension $\dim_{\mathbf{K}}(M) > 1$ over \mathbf{K} can be used in the membrane theory which is the generalization of the string (superstring) theory.

In Section 2 smashed products of wrap groups are constructed. Iterated wrap groups are studied as well. Their structure is investigated in more details than in the previous paper. Relations with path groups are studied. The main results of Section 2 are Theorems 1, 5, 7, 8, 17, 18, 20, Propositions 6, 10, 11, 14, 16 and Corollary 9.

All main results of this paper are obtained for the first time.

2 Structure of wrap groups.

Henceforth notations of the previous article are used [15].

1. Theorem. *Let M and N be C_β^α manifolds over an infinite field \mathbf{K} or a Cayley-Dickson algebra \mathcal{A}_r for \mathbf{K} , $1 \leq r \leq 3$, with a non-trivial multiplicative ultra-norm, where M and N are of dimensions over \mathbf{K} not less than one, $\dim_{\mathbf{K}} M \geq 1$ and $\dim_{\mathbf{K}} N \geq 1$. Then a wrap group $(W^M N)_{\alpha, \beta}$ has no any nontrivial continuous one parameter subgroup $\{g^b : b \in (\mathbf{K}, +)\}$.*

Proof. Each manifold over \mathcal{A}_r has a structure of a manifold over \mathbf{K} as well, hence it is sufficient to demonstrate this theorem for M and N over \mathbf{K} . We shall demonstrate that any non unit element, $g \neq e$, in $(W^M N)_{\alpha, \beta}$ does not belong to any one-parameter subgroup $\mathbf{K} \ni b \mapsto g^b$, where $(\mathbf{K}, +)$ is considered as an additive group. We already know that $(W^M N)_{\alpha, \beta}$ is the commutative group (see Theorem 6 [15]). As usually a continuous one parameter subgroup means a continuous homomorphism $\phi(b) = g^b$ so that $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in \mathbf{K}$, $\phi : \mathbf{K} \rightarrow (W^M N)_{\alpha, \beta}$ is continuous, that is $g^{a+b} = g^a + g^b$, when the group operation in the wrap group is denoted as the addition.

Suppose the contrary, that $\{g^b : b \in (\mathbf{K}, +)\}$ is a nontrivial continuous one parameter subgroup, that is, $g^b \neq e$ for $b \neq 0$. The element g is different from e , hence its equivalence class $g = \langle f \rangle_{K, \alpha, \beta}$ for some $f \in C_\beta^\alpha(\bar{M}, \{s_{0, q} :$

$q = 1, \dots, k\}$); (N, y_0) is different from $\langle w_0 \rangle_{K, \alpha, \beta}$, where $w_0(M) = \{y_0\}$, $s_{0,q}$ with $q = 1, \dots, k$ are marked points in \bar{M} and y_0 is a marked point in N , $M = \bar{M} \setminus \{s_{0,q} : q = 1, \dots, k\}$. In accordance with our convention the manifolds M and N are modeled on locally \mathbf{K} convex space X and Y over \mathbf{K} . Take a chart $(U_i^N, {}_N\phi_i)$ of N so that $y_0 \in U_i^N \subset N$ (see §2.3 in [15]). Its image $z_0 = {}_N\phi_i(y_0)$ belongs to Y . Without loss of generality we can consider that z_0 corresponds to zero in Y making a shift $Y \ni y \mapsto y - z_0 \in Y$ in a case of necessity. Therefore, there exists a continuous ultra-norm v on Y so that

$$(1) \quad t := \sup_{j \in \Lambda_N, k \in \Lambda_M; x \in M_{j,k}} v(f_{j,k}(x)) > 0,$$

where $f_{j,k}$ denotes a composition of f with chart mappings of the manifolds N and M respectively, $f_{j,k} = {}_N\phi_j \circ f \circ {}_M\phi_k^{-1}$, $M_{j,k}$ is a definition domain of $f_{j,k}$, $M_{j,k} \subset X$.

Let at first the field \mathbf{K} be of zero characteristic $\text{char}(\mathbf{K}) = 0$. Then there exists a prime number $p > 1$ so that $0 < |p|_{\mathbf{K}} < 1$, consequently, $\lim_{n \rightarrow \infty} p^n = 0$ in \mathbf{K} , where $n \in \mathbf{N}$. If g^b is continuous by b , then $\lim_{n \rightarrow \infty} g^{p^n} = 0$. On the other hand,

$$(2) \quad g^{p^n} = \langle \chi^*(f^{\vee p^n}) \rangle_{K, \alpha, \beta},$$

where $f^{\vee m}$ denotes the m fold wedge product of f with itself, χ^* is the homomorphism for an m fold wedge products of mappings from $M^{\vee m}$ into N induced by pairwise wedge products by induction from Theorem 3.3 [15] denoted here also by χ^* [15], $m \in \mathbf{N}$. But

$$(3) \quad \sup_{j \in \Lambda_N, k \in \Lambda_M; x \in M_{j,k}} v([\chi^*(f^{\vee p^n})]_{j,k}(x)) = t > 0$$

which contradicts (2), consequently, the supposition about an existence of a continuous nontrivial one parameter subgroup in $(W^M N)_{\alpha, \beta}$ was false.

If $\{g^b : b \in (\mathbf{K}, +)\}$ is a continuous non trivial one parameter subgroup and $\text{char}(\mathbf{K}) = z > 0$, then the z fold sum of the unit is zero, $1 + \dots + 1 = 0$, in the field \mathbf{K} . Thus we would have $g^z = g^0 = e$, since $g^0 + g^b = g^b$ for each $b \in \mathbf{K}$. But again

$$(4) \quad \sup_{j \in \Lambda_N, k \in \Lambda_M; x \in M} v([\chi^*(f^{\vee z})]_{j,k}(x)) = t > 0$$

that contradicts (2). Therefore, the theorem is proved in this case also.

2. Remark. Mention that apart from the classical case over the fields \mathbf{R} or \mathbf{C} the exponential function on non-archimedean fields has only finite radius of convergence and each differential equation with initial conditions generally have infinite families of solutions, because for example on the field \mathbf{Q}_p of p -adic numbers there is an infinite family of functions f not equal to constants or even different from locally constant functions, but with the

derivative $df(x)/dx$ equal to zero on \mathbf{Q}_p [22, 23].

It is also interesting to mention that under rather mild conditions a manifold M modeled on a non-archimedean Banach space X can be embedded into it as a clopen subset, $M \hookrightarrow X$ (see Reference [24] in [15]).

3. Iterated wrap groups. We denote by $(\mathcal{P}^M E)_{\alpha,\beta}$ a space of equivalence classes $\langle f \rangle_{K,\alpha,\beta}$ of $f \in C_\beta^\alpha(\bar{M}, \mathcal{W})$ relative to the closures of orbits of the left action of the family $Di_{\beta,0}^\alpha(M)$ defined in §3.2 [15] of all generalized diffeomorphisms of a $C_\beta^{\alpha'}$ differentiable space \bar{M} . This means, that $(\mathcal{P}^M E)_{\alpha,\beta}$ is the quotient space of $C_\beta^\alpha(\bar{M}, \mathcal{W})$ relative to the equivalence relation $K_{\alpha,\beta}$. It may be worthwhile to mention that in the particular case of $\alpha = 0$ the diffeomorphism group $Diff_{\beta,0}^\alpha(\bar{M})$ reduces to the homeomorphism group of preserving marked points mappings belonging to $C_{\beta,0}^0$ class.

On the other hand, there is the embedding $\theta : C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; \mathcal{W}, y_0) \hookrightarrow C_\beta^\alpha(\bar{M}; \mathcal{W})$ and the evaluation mapping $\hat{e}v : C_\beta^\alpha(\bar{M}; \mathcal{W}) \rightarrow N^k$ such that $\hat{e}v(f) := (\hat{f}(\hat{s}_{0,q}) : q = k+1, \dots, 2k)$, $\hat{e}v_{\hat{s}_{0,q}}(f) := \hat{f}(\hat{s}_{0,q})$, where $\hat{f} \in C_\beta^\alpha(\hat{M}; \mathcal{W})$ is such that $\hat{f} = f \circ \Xi$, $\Xi : \hat{M} \rightarrow \bar{M}$ is the quotient mapping. Thus we get the diagram $C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; \mathcal{W}, y_0) \rightarrow C_\beta^\alpha(\bar{M}; \mathcal{W}) \rightarrow N^k$ with C_β^α differentiable mappings, which induces the diagram ${}_{l+1}C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; \mathcal{W}, y_0) \rightarrow C_\beta^\alpha(\bar{M}, {}_l C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; \mathcal{W}, y_0)) \rightarrow {}_l C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; \mathcal{W}, y_0)$ for each $l \in \mathbf{N}$, where ${}_{l+1}C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; \mathcal{W}, y_0) := C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; {}_l C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; \mathcal{W}, y_0))$, also ${}_1 C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; \mathcal{W}, y_0) := C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; \mathcal{W}, y_0)$. Using this procedure we get iterated wrap semigroups and groups $(S^M E)_{l+1;\alpha,\beta} := (S^M (S^M E)_{l;\alpha,\beta})_{\alpha,\beta}$ and $(W^M E)_{l+1;\alpha,\beta} := (W^M (W^M E)_{l;\alpha,\beta})_{\alpha,\beta}$, where $(S^M E)_{1;\alpha,\beta} := (S^M E)_{\alpha,\beta}$ and $(W^M E)_{1;\alpha,\beta} := (W^M E)_{\alpha,\beta}$.

Evidently, if there are C_β^α and $C_\beta^{\alpha'}$ diffeomorphisms $\rho : \bar{M} \rightarrow \bar{M}_1$ and $\eta : N \rightarrow N_1$ mapping marked points into respective marked points, then $C_\beta^\alpha(\bar{M}, \mathcal{W})$ is isomorphic with $C_\beta^\alpha(\bar{M}_1, \mathcal{W}_1)$ and hence $(W^M E)_{v;\alpha,\beta}$ is C_β^α isomorphic as the C_β^α differentiable space and C_β^l -isomorphic as the C_β^l differentiable Lie group with $(W^{M_1} E_1)_{v;\alpha,\beta}$ for each $v \in \mathbf{N}$, where $l = \alpha' - \alpha$, $l = \infty$ for $\alpha' = \infty$, $\alpha' \geq \alpha \geq 0$. In particular, if M, N, G, E are $C_\beta^{\alpha'}$ manifolds, then $(W^M E)_{\alpha,\beta}$ is a C_β^α manifold (see §§3.3, 3.6 [15]). If $f : N \rightarrow N_1$ is a surjective map and N_1 is a C_β^α -differentiable space, then N inherits a structure of an C_β^α -differentiable space with plots having the local form $f \circ \rho : U \rightarrow N_1$, where $\rho : U \rightarrow N$ is a plot of N .

4. Lemma. *Let E be a $C_\beta^{\alpha'}$ principal bundle and let D be an every-*

where dense subset in N such that for each $y \in D$ there exists an open neighborhood V of y in N and a differentiable map $p : V \rightarrow C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; V, y) := \{f \in C_\beta^\alpha(\bar{M}; V) : f(s_{0,q}) = y, q = 1, \dots, k\}$ such that $\hat{e}v_{\hat{s}_{0,q}}(\hat{p}(y)) = y$ for each $q = 1, \dots, 2k$ and each $y \in N$, where $p \circ \Xi = \pi \circ \hat{p}$. Then $\hat{e}v : C_\beta^\alpha(\bar{M}; \mathcal{W}) \rightarrow N^k$ is a C_β^α differentiable principal $(S^M E)_{\alpha,\beta}$ bundle.

Proof. Let $\{(V_j, y_j) : j \in J\}$ be a family such that $y_j \in V_j \cap D$ for each j and there exists a mapping $p_j : V_j \rightarrow C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; V_j, y_j)$ so that $\hat{p}_j(\hat{s}_{0,q})(y) = y \times e$ for each $q = 1, \dots, 2k$ and every j , where $\{V_j : j \in J\}$ is an open covering of N , y is a constant mapping from \hat{M} into V_j with $y(\hat{M}) = \{y\}$, where $\hat{p}_j(\hat{s}_{0,q})$ is the restriction to V_j of the projection $\hat{p}(\hat{s}_{0,q}) : (\mathcal{P}^M E)_{\alpha,\beta} \rightarrow E$, while $p_j(\Xi(\hat{x}))(y) = \pi \circ \hat{p}_j(\hat{x})(y \times e)$ for each $y \in N$ and $x = \Xi(\hat{x})$ in \bar{M} , where $\hat{x} \in \hat{M}$, $\Xi : \hat{M} \rightarrow \bar{M}$. Then $(W^M E)_{\alpha,\beta}$ and $(\mathcal{P}^M E)_{\alpha,\beta}$ are supplied with the C_β^α -differentiable spaces structure (see Remark 3 above and Theorem 6 in [15]), where the embedding $(S^M E)_{\alpha,\beta} \hookrightarrow (\mathcal{P}^M E)_{\alpha,\beta}$ and the projection $\hat{e}v_{\hat{s}_{0,q}} : (\mathcal{P}^M E)_{\alpha,\beta} \rightarrow N$ are C_β^α -maps.

Let a generalized diffeomorphism $\psi_j \in Di_\beta^\alpha(N)$ be such that $\psi_j(y) = y_j$. Specify a trivialization $\phi_j : \hat{p}_j^{-1}(\hat{s}_{0,q})(V_j) \rightarrow V_j \times (S^M E)_{\alpha,\beta}$ of the restriction $\hat{p}_j(\hat{s}_{0,q})|_{V_j}$ of the projection $\hat{p}_j(\hat{s}_{0,q}) : (\mathcal{P}^M E)_{\alpha,\beta} \rightarrow E$ by the formula $\phi_j(f) = (f(\hat{s}_{0,q}), \psi_j \circ \hat{p}_j(\hat{s}_{0,q})(f))$ for each $f \in (\mathcal{P}^M E)_{\alpha,\beta}$ with $\pi \circ f(\hat{s}_{0,q}) = y$, where $\psi_j \circ \hat{p}_j(f) = \psi_j(\hat{p}_j(f))$. Then $\phi_j^{-1}(y, g) = g^{-1}(\psi_j \circ \hat{p}_j(y)) =: \eta$, $\eta \in (\mathcal{P}^M E)_{\alpha,\beta}$ with $\pi \circ \psi_j \circ f(\hat{s}_{0,q}) = y_j$, since G is a group, where $g = \psi_j \circ \hat{p}_j(f)$. A combining of the family $\{\hat{e}v_{\hat{s}_{0,q}} : q = k+1, \dots, 2k\}$ induces a mapping $\hat{e}v : C_\beta^\alpha(\bar{M}; \mathcal{W}) \rightarrow N^k$. By the construction above a fiber of this bundle is the monoid $(S^M E)_{\alpha,\beta}$.

5. Theorem. If \bar{M} and N are C_β^α manifolds over \mathcal{A}_r , $0 \leq r \leq 3$, \hat{M} and N are embedded as clopen absolutely \mathcal{A}_r convex subsets into Banach spaces Z_M and Z_N over \mathcal{A}_r , then there exists a C_β^α -differentiable principal $(S^M E)_{\alpha,\beta}$ bundle $\hat{e}v : (\mathcal{P}^M E)_{\alpha,\beta} \rightarrow N^k$.

Proof. In accordance with Lemma 4 it is sufficient to prove that for each $y \in N$ there exist a neighborhood U of y in N and a C_β^α -map $p_q : U \rightarrow C_\beta^\alpha(\bar{M}, \mathcal{W})$ such that $\hat{e}v_{s_{0,q}}(p_q(z)) = z$ for each $q = 1, \dots, k$, $z \in U$, where $\hat{e}v_x(f) = f(x)$.

For \hat{M} consider \mathbf{K} linear mappings $\zeta_q : B(\mathbf{K}, 0, 1) \rightarrow \hat{M}$ joining $\hat{s}_{0,q}$ with $\hat{s}_{0,q+k}$ such that $\zeta_q(0) = \hat{s}_{0,q}$ and $\zeta_q(1) = \hat{s}_{0,q+k}$, since \hat{M} is embedded as a clopen absolutely \mathcal{A}_r convex subset into a Banach space Z_M over \mathcal{A}_r , where $1 \leq q \leq k$, $B(Z, z, R) := \{y \in Z : \rho(y, z) \leq R\}$ denotes a ball of radius

$0 < R$ with a center at z in a metric space Z with a metric ρ . We consider a coordinate system (x_1, \dots, x_m, \dots) in \hat{M} over \mathbf{K} such that x_1 corresponds to a natural coordinate along ζ_q . This coordinate system is defined globally for a chosen q .

The manifold N is also embedded as the clopen absolutely \mathcal{A}_r convex subset into a Banach space Z_N over \mathcal{A}_r . Then for each chart U in N there exists a map $p_q : U \rightarrow (\mathcal{P}^M U)_{\alpha, \beta}$ with $\pi \circ [p_q(\hat{s}_{0, q+k})(z)] = z$ and $\pi \circ [p_q(\hat{s}_{0, q})(z)] = y$ for each $z \in U$, where $p_q \circ \zeta_q =: \hat{\gamma}_{q, y, z}$ is a mapping so that $\pi \circ \hat{\gamma}_{q, y, z}$ is a \mathbf{K} linear mapping in U joining y with z , $\hat{\gamma}_{q, y, z} : B(\mathbf{K}, 0, 1) \rightarrow N$, $\hat{\gamma}_{q, y, z} \circ \zeta_q^{-1}(x_1) \in N$ for each x_1 , where $\pi : E \rightarrow N$ is the projection of the fiber bundle. Having initially $\hat{\gamma}_{q, y, z}$ we extend it to \hat{p}_q on \hat{M} with values in E such that $p_q \circ \Xi = \pi \circ \hat{p}_q$.

6. Proposition. (1). *The wrap group $(W^M E; N, G, \mathbf{P})_{\alpha, \beta}$ has the a structure of a principal G^k bundle over $(W^M N)_{\alpha, \beta}$ if either M and N satisfy conditions of Theorem 5 or G^k acts effectively on $(W^M E)_{\alpha, \beta}$.*

(2). *The abelianization $[(W^M E; N, G, \mathbf{P})_{\alpha, \beta}]_{ab}$ of the wrap group $(W^M E; N, G, \mathbf{P})_{\alpha, \beta}$ is isomorphic with $(W^M E; N, G_{ab}, \mathbf{P})_{\alpha, \beta}$.*

Proof. 1. We have the bundle structure $\pi : E \rightarrow N$. It induces the bundle structure $\hat{\pi} : (W^M E; N, G, \mathbf{P})_{\alpha, \beta} \rightarrow (W^M N)_{\alpha, \beta}$, since $\pi \circ \mathbf{P}_{\hat{\gamma}, u} = \hat{\gamma}$. In accordance with Lemma 4 it is sufficient to show, that there exists a neighborhood U_G of e in $(W^M E)_{\alpha, \beta}$ and a G -equivariant mapping $\phi : U_G \rightarrow (W^M N)_{\alpha, \beta}$ (see Conditions 3.2(P1–P4) [15]). Let $\langle \mathbf{P}_{\hat{\gamma}, u} \rangle_{\alpha, \beta} \in (W^M E)_{\alpha, \beta}$, where $\hat{\gamma} : \hat{M} \rightarrow N$, $\hat{\gamma} = \gamma \circ \Xi$, $\gamma : \bar{M} \rightarrow N$, $\gamma(s_{0, q}) = y_0$ for each $q = 1, \dots, k$. Then $\pi \circ \mathbf{P}_{\hat{\gamma}, u} = \hat{\gamma}$ and $\mathbf{P}_{\hat{\gamma}, u}$ is G -equivariant by the conditions defining the parallel transport structure. This means that $\mathbf{P}_{\hat{\gamma}, u}(x)z = \mathbf{P}_{\hat{\gamma}, uz}(x)$ for each $x \in \hat{M}$ and $z \in G$ and every $u \in E_{y_0}$. We have that $uG = \pi^{-1}(y)$ for each $u \in E_y$ and $y \in N$.

Therefore, put $\phi = \pi_*$, where $\pi_* \langle \mathbf{P}_{\hat{\gamma}, u} \rangle_{\alpha, \beta} = \langle \hat{\gamma}, u \rangle_{\alpha, \beta}$ and take $U_G = \pi_*^{-1}(U)$, where U is a symmetric $U^{-1} = U$ neighborhood of e in $(W^M N)_{\alpha, \beta}$.

The group G acts effectively on E . Then G^k acts effectively on $(W^M E)_{\alpha, \beta}$ if N and M satisfy conditions of Theorem 5. Indeed, for each ζ_q from §5 there is $g_q \in G$ corresponding to $\hat{\gamma}(\hat{s}_{0, q+k})$ with $\mathbf{P}_{\hat{p}_q, \hat{s}_{0, q} \times e}(\hat{s}_{0, q+k}) = \{y_0 \times g_q\} \in E_{y_0}$, $g_q \in G$ for every $q = 1, \dots, k$. Moreover, $\pi_*^{-1}(\pi_*(\langle \mathbf{P}_{\hat{\gamma}, u} \rangle_{\alpha, \beta})) = \langle \mathbf{P}_{\hat{\gamma}, u} \rangle_{\alpha, \beta}$ G^k . Then the fibre of $\hat{\pi} : (W^M E; N, G, \mathbf{P})_{\alpha, \beta} \rightarrow (W^M N)_{\alpha, \beta}$ is G^k . Due to Conditions 2(P1–P4) in Section 3.2 [15] it is the principal G^k differentiable bundle of class C_β^α .

2. Therefore, due to $\dim_{\mathbf{K}} N \geq 1$ the considered here wrap groups are infinite dimensional over \mathbf{K} . Thus Statement (2) follows from the proof of Theorem 5 above, since the wrap groups $(W^M N)_{\alpha, \beta}$ for $G = \{e\}$ and $(W^M E; N, G_{ab}, \mathbf{P})_{\alpha, \beta}$ for $G = G_{ab}$ are commutative (see Theorem 6(2) [15]).

7. Theorem. *Let $\text{Diff}_{\beta}^{\alpha'}(N)$ act transitively on N , $\alpha \leq \alpha'$, where M and N are embedded as clopen \mathcal{A}_r absolutely convex subsets into Banach spaces Z_M and Z_N over \mathcal{A}_r . For each C^∞ manifold N and an C_β^α differentiable group G such that $\mathcal{A}_r^* \subset G$ with $1 \leq r \leq 3$ there exists a homomorphism of the C_β^α differentiable space of all equivalence classes of $(\mathcal{P}^M E)_{\alpha, \beta}$ relative to $\text{Diff}_{\beta}^{\alpha'}(N)$ (see §§1 and 2 in Section 3 [15] and §3 above) into $\text{Hom}_{\beta}^{\alpha}((S^M E)_{\alpha, \beta}, G^k)$. They are isomorphic, when G is commutative.*

Proof. Mention that due to Theorem 5 the C_β^α -differentiable principal $(S^M E)_{\alpha, \beta}$ bundle $\hat{e}v : (\mathcal{P}^M E)_{\alpha, \beta} \rightarrow N^k$ has a parallel transport structure $\hat{\mathbf{P}}_{\hat{\gamma}, uz}(x) = \hat{\mathbf{P}}_{\hat{\gamma}, u}(x)z$ for each $x \in \hat{M}$ and all $\gamma \in C_\beta^\alpha(\bar{M}, N)$ and $u \in \hat{e}v^{-1}(\gamma(s_{0,k}))$ and every $z \in G$ and the corresponding $\hat{\gamma} : \hat{M} \rightarrow N$ such that $\gamma \circ \Xi = \hat{\gamma}$. If $x = \hat{s}_{0,q}$ with $1 \leq q \leq k$, then $\hat{\mathbf{P}}$ gives the identity homomorphism from $(S^M E)_{\alpha, \beta}$ into $(S^M E)_{\alpha, \beta}$. If $\theta : (S^M E)_{\alpha, \beta} \rightarrow G^k$ is an C_β^α differentiable homomorphism, then the holonomy of the associated parallel transport $\hat{\mathbf{P}}^\theta$ on the bundle $(\mathcal{P}^M E)_{\alpha, \beta} \times^\theta G \rightarrow N^k$ is the homomorphism $\theta : (S^M E)_{\alpha, \beta} \rightarrow G^k$ (see §6 in Section 3 [15]). At the same time the group G contains continuous multiplicative one-parameter subgroups from \mathcal{A}_r^* , where $1 \leq r \leq 3$. If $g \in (W^M N)_{\alpha, \beta}$ and $g \neq e$, then g is of infinite order, since w_0 does not belong to g^n for each $n \neq 0$ non-zero integer n , where $w_0(M) = \{y_0\}$.

This holonomy induces a map $h : (\mathcal{P}^M E)_{\alpha, \beta} / \mathcal{Q} \rightarrow \text{Hom}_{\beta}^{\alpha}((S^M E)_{\alpha, \beta}, G^k)$ with values in the family of homomorphisms of class C_β^α from $(S^M E)_{\alpha, \beta}$ into G^k , where \mathcal{Q} is an equivalence relation caused by the transitive action of $\text{Diff}_{\beta}^{\alpha'}(N)$ such that $(S^M E)_{\alpha, \beta}$ with distinct marked points $\{s_{0,q} : q = 1, \dots, k\}$ in M and either y_0 or \tilde{y}_0 in N are isomorphic, since there exists $\psi \in \text{Diff}_{\beta}^{\alpha'}(N)$ such that $\psi(y_0) = \tilde{y}_0$.

If G is commutative, then this map is the homomorphism, since $(S^M E)_{\alpha, \beta}$ is the commutative monoid for a commutative group G (see Theorem 3 in Section 3 [15]) and $u\mathbf{P}_{\hat{\gamma}_1, v_1}(x_1)\mathbf{P}_{\hat{\gamma}_2, v_2}(x_2) = u\mathbf{P}_{\hat{\gamma}_2, v_2}(x_2)\mathbf{P}_{\hat{\gamma}_1, v_1}(x_1)$ for each $x_1, x_2 \in \hat{M}$ and $u, v_1, v_2 \in E_{y_0}$. We have the embedding $(S^M E)_{\alpha, \beta} \hookrightarrow (W^M E)_{\alpha, \beta}$. Thus a homomorphism $\theta : (W^M E)_{\alpha, \beta} \rightarrow G^k$ has the restriction on $(S^M E)_{\alpha, \beta}$ which is also the homomorphism.

For $G \supset \mathcal{A}_r^*$ there exists a family of $f \in \text{Hom}_\beta^\alpha((S^M E)_{\alpha,\beta}, G^k)$ separating elements of the wrap monoid $(S^M E)_{\alpha,\beta}$, hence there exists the embedding of $(S^M E)_{\alpha,\beta}$ into $\text{Hom}_\beta^\alpha((S^M E)_{\alpha,\beta}, G^k)$. The bundle $(\mathcal{P}^M E)_{\alpha,\beta} \times^\theta G \rightarrow N^k$ has the induced parallel transport structure \mathbf{P}^θ . The holonomy of the parallel transport structure on $(\mathcal{P}^M N)_{\alpha,\beta} \times^\theta G \rightarrow N^k$ is θ . Therefore, the map $C_\beta^\alpha((S^M E)_{\alpha,\beta}, G^k) \ni \theta \mapsto \mathbf{P}^\theta$ is inverse to h .

8. Embeddings of wrap groups and normal subgroups. Suppose that

(E1) there are embeddings $\bar{M}_2 \hookrightarrow \bar{M}_1$ and $\bar{M} = \bar{M}_1 \setminus (\bar{M}_2 \setminus \bar{M}_{2,f})$ and $\hat{M}_2 \hookrightarrow \hat{M}_1$ and $\hat{M} = \hat{M}_1 \setminus (\hat{M}_2 \setminus \hat{M}_{2,f})$ and $N_2 \hookrightarrow N_1$ for C_β^α -manifolds with the same marked points $\{s_{0,q} : q = 1, \dots, k\}$ for \bar{M}_1 and \bar{M}_2 and \bar{M} and $y_0 \in N_2$ satisfying conditions of §§1 and 2 in Section 3 [15] and

(E2) G_2 is a closed subgroup in G_1 with a complete relative to its uniformity principal fiber bundle E with a structure group G_1 . Moreover, we suppose that atlases of all embedded pairs of manifolds $A \hookrightarrow B$ are consistent in the following sense.

(E3) Each chart U of A is contained in some chart V of B so that there exists a C_β^α embedding $\theta_{U,V} : \phi_U(U) \hookrightarrow \phi_V(V) \subset X_B$, where

(E4) $\phi_U : U \rightarrow \phi_U(U)$ and $\phi_V : V \rightarrow \phi_V(V)$ are homeomorphisms,

(E5) $\phi_U(U)$ and $\phi_V(V)$ are \mathbf{K} convex in X_B , where X_B is a complete locally \mathbf{K} convex space. Moreover,

(E6) there exists a topological Schauder basis $\{e_j : j \in J\}$ in X_B , where J is a set. Suppose also that

(E7) for each point $x_0 \in \theta_{U,V}(\phi_U(U))$ either $(x_0 + e_j \mathbf{K}) \cap \theta_{U,V}(\phi_U(U))$ is clopen in $\phi_V(V) \cap (x_0 + e_j \mathbf{K})$ or a singleton is. That is this consistency of atlases is satisfied for pairs (\bar{M}_2, \bar{M}_1) , (\hat{M}_2, \hat{M}_1) and (N_2, N_1) . Suppose also that

(E8) manifolds \hat{M}_1 and \bar{M}_1 are finite dimensional over \mathbf{K} .

Theorem. 1. *Then $(W^{M_2, \{s_{0,q}:q=1,\dots,k\}} E; N_2, G_2, \mathbf{P})_{\alpha,\beta}$ has an embedding as a closed subgroup into $(W^{M_1, \{s_{0,q}:q=1,\dots,k\}} E; N_1, G_1, \mathbf{P})_{\alpha,\beta}$.*

2. *The wrap group $(W^{M_2, \{s_{0,q}:q=1,\dots,k\}} E; N, G_2, \mathbf{P})_{\alpha,\beta}$ is normal in $(W^{M_1, \{s_{0,q}:q=1,\dots,k\}} E; N, G_1, \mathbf{P})_{\alpha,\beta}$ if and only if G_2 is a normal subgroup in G_1 .*

3. *In the latter case $(W^M E; N, G, \mathbf{P})_{\alpha,\beta}$ is isomorphic with $(W^{M_1} E; N, G_1, \mathbf{P})_{\alpha,\beta} / (W^{M_2} E; N, G_2, \mathbf{P})_{\alpha,\beta}$, where $G = G_1 / G_2$.*

Proof. 1. Manifolds \bar{M}_1 and \bar{M}_2 are finite dimensional over the field \mathbf{K} .

Each finite dimensional vector space over the field \mathbf{K} is isomorphic with \mathbf{K}^n for some natural number $n \in \mathbf{N}$. Therefore, \bar{M}_1 and \bar{M}_2 have disjoint clopen atlases refining their initial atlases. Without loss of generality we can take such atlases.

We recall that a system of \mathbf{K} linearly independent vectors $\{e_j : j \in J\}$ is called a topological basis in a topological \mathbf{K} vector space X , if each $x \in X$ can be decomposed as a limit of finite \mathbf{K} linear combinations of elements e_j with components $a_j = \zeta_j(x) \in \mathbf{K}$ of x , where each $\zeta_j(x)$ is a \mathbf{K} linear functional on X , $x = \lim \sum_j \zeta_j(x) e_j$. A topological basis $\{e_j : j \in J\}$ with continuous \mathbf{K} linear functionals $\zeta_j : X \rightarrow \mathbf{K}$ for each $j \in J$ is called a Schauder basis of X .

For manifolds \hat{M}_1 and \bar{M}_1 a vector space X on which they are modeled is finite dimensional over \mathbf{K} by the supposition of this theorem.

Moreover, $\phi_U(U)$ can be presented as a disjoint union $\phi_U(U) = \bigcup_j A_j$ of balls clopen in $x_j + \text{span}_{\mathbf{K}}\{e_{k_1(j)}, \dots, e_{k_m(j)}\}$, where $m = n_2$ is a dimension of $\phi_U(U)$ over \mathbf{K} , $k_1(j), \dots, k_m(j)$ are pairwise different natural numbers in $\{1, \dots, n_1\}$, n_1 is a dimension of $\phi_V(V)$ over \mathbf{K} , since $\phi_U(U)$ and $\phi_V(V)$ satisfy Conditions (E3 – E8).

Each complete locally \mathbf{K} convex space Y is a projective limit of Banach spaces Y_ξ over \mathbf{K} , where $\xi \in \Lambda_Y$, Λ_Y is a directed set [18]. If Y has a topological Schauder basis, then each Y_ξ has a topological Schauder basis $\{e_{j,\xi} : j \in J_\xi\}$, where J_ξ is a set. A function $f : \hat{M}_1 \rightarrow Y$ is of C_β^α class if and only if $\pi_\xi \circ f : \hat{M}_1 \rightarrow Y_\xi$ is of C_β^α class for each ξ , where $\pi_\xi : Y \rightarrow Y_\xi$ is the quotient mapping. Pointwise we have a decomposition $\pi_\xi \circ f(x) = \lim \sum_j f_{j,\xi}(x) e_{j,\xi}$, where $f_{j,\xi}(x) = \zeta_{j,\xi}(f(x))$, $x \in \hat{M}_1$, $\zeta_{j,\xi} = \pi_\xi \circ \zeta_j$. If each $f_{j,\xi}$ is of C_β^α class, then $\pi_\xi \circ f$ is of C_β^α class.

If $f : T \rightarrow \mathbf{K}$ is a C_β^α function from a clopen subset T in \mathbf{K}^m into \mathbf{K} or X_N , then it has a C_β^α extension on \mathbf{K}^m taking $f|_{(\mathbf{K}^m \setminus T)} = g$, where g is a C_β^α function on $\mathbf{K}^m \setminus T$, since the latter set is also clopen in \mathbf{K}^m (see §2 in Section 2 [15]). If T is a singleton in \mathbf{K} , then evidently a locally constant or some other C_β^α extension from T on \mathbf{K} exists.

Thus we get C_β^α extensions from A_j on B_j , where B_j is clopen in X_{M_1} , $B_j \cap (x_j + \text{span}_{\mathbf{K}}\{e_{k_1(j)}, \dots, e_{k_m(j)}\}) = \theta_{U,V}(A_j)$ for each j , $\bigcup_j B_j = \phi_V(V)$, due to Theorem 40 [11] for the finite dimensional over \mathbf{K} space X for functions with values in X_{N_1} , where X_{N_1} is a \mathbf{K} vector space on which N_1 is modeled. Combining these disjoint clopen coverings and mappings on them we get a

C_β^α extension from M_2 onto M_1 . Therefore, if $\hat{\gamma}_2 \in C_\beta^\alpha(\hat{M}_2, N_2)$, then it has a C_β^α extension to $\hat{\gamma}_1 \in C_\beta^\alpha(\hat{M}_1, N_1)$, since these manifolds are totally disconnected and their atlases are consistent.

Thus the parallel transport structure $\mathbf{P}_{\hat{\gamma}_1, u}$ over \hat{M}_1 serves as an extension of $\mathbf{P}_{\hat{\gamma}_2, u}$ over \hat{M}_2 . The uniform spaces $C_\beta^\alpha(\bar{M}_j, \{s_{0,1}, \dots, s_{0,k}\}; \mathcal{W}_j, y_0)$ are complete for $j = 1, 2$, since the principal fiber bundle E is complete relative to its uniformity and the corresponding principal fiber sub-bundle E_2 with the structure group G_2 is also complete (see Theorem 8.3.6 [3]). Therefore, $C_\beta^\alpha(\bar{M}_2, \{s_{0,1}, \dots, s_{0,k}\}; \mathcal{W}_2, y_0)$ has embedding as the closed subspace into $C_\beta^\alpha(\bar{M}_1, \{s_{0,1}, \dots, s_{0,k}\}; \mathcal{W}_1, y_0)$. Using Theorem 40 [11] as above we infer that each C_β^α diffeomorphism of \bar{M}_2 has an C_β^α extension to a diffeomorphism of \bar{M}_1 . From the condition that G_2 is a closed subgroup in G_1 we infer that $(S^{M_2, \{s_{0,q}:q=1,\dots,k\}} E; N_2, G_2, \mathbf{P})_{\alpha, \beta}$ has an embedding as a closed sub-monoid into $(S^{M_1, \{s_{0,q}:q=1,\dots,k\}} E; N_1, G_1, \mathbf{P})_{\alpha, \beta}$ and inevitably $(W^{M_2, \{s_{0,q}:q=1,\dots,k\}} E; N_2, G_2, \mathbf{P})_{\alpha, \beta}$ has an embedding as a closed subgroup into $(W^{M_1, \{s_{0,q}:q=1,\dots,k\}} E; N_1, G_1, \mathbf{P})_{\alpha, \beta}$ due to Theorem 6.1 in Section 3 [15].

2. The groups $(W^{M_j, \{s_{0,q}:q=1,\dots,k\}} N)_{\alpha, \beta}$ for $j = 1, 2$ are commutative and $(W^{M_j, \{s_{0,q}:q=1,\dots,k\}} E)_{\alpha, \beta}$ is the G_j^k principal fiber bundle on $(W^{M_j, \{s_{0,q}:q=1,\dots,k\}} N)_{\alpha, \beta}$ (see Theorem 6.2 in Section 3 [15] and Proposition 6.1 above). Therefore, $(W^{M_2, \{s_{0,q}:q=1,\dots,k\}} E)_{\alpha, \beta}$ is the normal subgroup in $(W^{M_1, \{s_{0,q}:q=1,\dots,k\}} E)_{\alpha, \beta}$ if and only if G_2 is the normal subgroup in G_1 .

3. Consider the principal fiber bundle $E(N, G, \pi, \Psi)$ with the structure group G (see Section 2.6 [15]) and the parallel transport structure \mathbf{P} for the C_β^α pseudo-manifold \hat{M} , where $G = G_1/G_2$ is the quotient group. If $\hat{\gamma}_1 \in C_\beta^\alpha(\hat{M}_1, N)$, then $\hat{\gamma}_1$ is the combination

$$(i) \hat{\gamma}_1 = \hat{\gamma}_2 \nabla \hat{\gamma},$$

where $\hat{\gamma}_2$ and $\hat{\gamma}$ are restrictions of $\hat{\gamma}_1$ on \hat{M}_2 and \hat{M} respectively. We also have that each $\hat{\gamma} \in C_\beta^\alpha(\hat{M}, N)$ has an extension $\hat{\gamma}_1 \in C_\beta^\alpha(\hat{M}_1, N)$. The manifold \hat{M}_1 is metrizable by a metric ρ . For each $\epsilon > 0$ there exists $\psi \in Dif_{\beta, 0}^\alpha(\hat{M}_1)$ such that $(\psi(\hat{M}) \cap \hat{M}_2) \subset \bigcup_{l=1}^s B(\hat{M}_1, x_l, \epsilon)$ for some $x_l \in \hat{M}_1$ with $l = 1, \dots, s$ and $s \in \mathbf{N}$ and $\psi|_{\hat{M}_1 \setminus (\hat{M} \bigcup_{l=1}^s B(\hat{M}_1, x_l, \epsilon_l))} = id$, since \hat{M}_1 and \hat{M}_2 are finite dimensional manifolds, where $B(\hat{M}_1, x, R)$ denotes a ball in \hat{M}_1 containing a point x and of radius $0 < R$, $0 < \epsilon_l < \infty$. Therefore, a using charts of the manifolds gives

$$< \mathbf{P}_{\hat{\gamma}, u}|_M >_{\alpha, \beta} = < \mathbf{P}_{\hat{\gamma}_1, u}|_{M_1} >_{\alpha, \beta} / < \mathbf{P}_{\hat{\gamma}_2, u}|_{M_2} >_{\alpha, \beta}$$

due to decomposition (i), since $\mathbf{P}_{\hat{\gamma},u}|_{M_j} \in G_j$ for $j = 1, 2$ and $G = G_1/G_2$ is the $C_\beta^{\alpha'}$ quotient group with $\alpha' \geq \alpha$. Consequently, $(W^M E; N, G, \mathbf{P})_{\alpha,\beta}$ is isomorphic with

$(W^{M_1} E; N, G_1, \mathbf{P})_{\alpha,\beta} / (W^{M_2} E; N, G_2, \mathbf{P})_{\alpha,\beta}$ (see also §§3, 6 in Section 3 [15]).

9. Corollary. *Let suppositions of Theorem 8 be satisfied. Then the group $(W^M N)_{\alpha,\beta}$ is isomorphic with the quotient group $(W^{M_1} N)_{\alpha,\beta} / (W^{M_2} N)_{\alpha,\beta}$.*

Proof. For $(W^M N)_{\alpha,\beta}$ taking $G = G_1 = G_2 = \{e\}$ we get the statement of this corollary from Theorem 8.3.

10. Proposition. *Suppose that $\bar{M} = \bar{M}_1 \vee \bar{M}_2$, where \bar{M}_1 and \bar{M}_2 are C_β^α differentiable spaces satisfying Conditions 2.6, 3.1 and 3.2 [15] with the bunch taken by marked points $\{s_{0,q} : q = 1, \dots, k\}$, then $(W^M N)_{\alpha,\beta}$ is isomorphic with the internal direct product $(W^{M_1} N)_{\alpha,\beta} \otimes (W^{M_2} N)_{\alpha,\beta}$.*

Proof. The C_β^α differentiable space \bar{M} has marked points $\{s_{0,q} : q = 1, \dots, k\}$ such that $s_{0,q}$ corresponds to $s_{0,q,1}$ glued with $s_{0,q,2}$ in the bunch $\bar{M}_1 \vee \bar{M}_2$ for each $q = 1, \dots, k$, where $s_{0,q,j} \in \bar{M}_j$ are marked points $j = 1, 2$. Each \bar{M}_j satisfies Conditions 3.1(S1 – S5) and 3.2 in [15], then M satisfies them also. Each $C_{\beta,0}^{\alpha,w_0}$ function on \bar{M}_1 in \bar{M} has a $C_{\beta,0}^{\alpha,w_0}$ extension as w_0 on M_2 . Due to the initial conditions at marked points $s_{0,q}$, $q = 1, \dots, k$, it has a $C_{\beta,0}^{\alpha,w_0}$ extension on \bar{M}_2 and thus on the entire \bar{M} also. Therefore, quite analogously to §8 $(W^{M_j, \{s_{0,q}:q=1,\dots,k\}} N)_{\alpha,\beta}$ has an embedding as a closed subgroup into $(W^{M, \{s_{0,q}:q=1,\dots,k\}} N)_{\alpha,\beta}$ for $j = 1, 2$. If $\gamma_j \in C_\beta^\alpha((\bar{M}_j, \{s_{0,q} : q = 1, \dots, k\}); (N, y_0))$ for $j = 1, 2$, then $\gamma_1 \vee \gamma_2 \in C_\beta^\alpha((\bar{M}, \{s_{0,q} : q = 1, \dots, k\}); (N, y_0))$. At the same time each $\gamma \in C_\beta^\alpha((\bar{M}, \{s_{0,q} : q = 1, \dots, k\}); (N, y_0))$ has the decomposition $\gamma = \gamma_1 \vee \gamma_2$, where $\gamma_j = \gamma|_{\bar{M}_j}$ for $j = 1, 2$. Therefore, $\langle \gamma \rangle_{\alpha,\beta} = \langle \gamma_1 \vee w_{0,2} \rangle_{\alpha,\beta} \vee \langle w_{0,1} \vee \gamma_2 \rangle_{\alpha,\beta}$, where $w_0(M) = \{y_0\}$, $w_{0,j} = w_0|_{M_j}$ for $j = 1, 2$, hence $(W^M N)_{\alpha,\beta}$ is isomorphic with $(W^{M_1} N)_{\alpha,\beta} \otimes (W^{M_2} N)_{\alpha,\beta}$.

11. Propositions. 1. *Let $\theta : N_1 \rightarrow N$ be an embedding with $\theta(y_1) = y_0$, or $F : E_1 \rightarrow E$ be an embedding of principal fiber bundles over \mathcal{A}_r such that $\pi \circ F|_{N_1 \times e} = \theta \circ \pi_1$, then there exist embeddings $\theta_* : (W^M N_1)_{\alpha,\beta} \rightarrow (W^M N)_{\alpha,\beta}$ and $F_* : (W^M E_1)_{\alpha,\beta} \rightarrow (W^M E)_{\alpha,\beta}$.*

2. *If $\theta : N_1 \rightarrow N$ and $F : E_1 \rightarrow E$ are a quotient mapping and a quotient homomorphism such that N_1 is a covering C_β^α differentiable space of a C_β^α differentiable space N satisfying conditions of §2.6 [15], then $(W^M N)_{\alpha,\beta}$ is the quotient group of some closed subgroup in $(W^M N_1)_{\alpha,\beta}$ and $(W^M E)_{\alpha,\beta}$ is the quotient group of some closed subgroup in $(W^M E_1)_{\alpha,\beta}$.*

3. If there are a C_β^α diffeomorphism $f_1 : M \rightarrow M_1$ and an $C_\beta^{\alpha'}$ -isomorphism $f_2 : E \rightarrow E_1$, then wrap groups $(W^{M_1}E_1)_{\alpha,\beta}$ and $(W^ME)_{\alpha,\beta}$ are isomorphic.

Proof. 1. If $\gamma_1 \in C_\beta^\alpha((\bar{M}, \{s_{0,q} : q = 1, \dots, k\}); (N_1, y_1))$, then $\theta \circ \gamma_1 = \gamma \in C_\beta^\alpha((\bar{M}, \{s_{0,q} : q = 1, \dots, k\}); (N, y_0))$, $\langle \gamma \rangle_{\alpha,\beta} = \theta_* \langle \gamma_1 \rangle_{\alpha,\beta}$, where $\theta_* \langle \gamma_1 \rangle_{\alpha,\beta} := \{\theta \circ f : f K_{\alpha,\beta} \gamma_1\}$. In addition $F|_{E_{1,v}}$ gives an embedding $F : G_1 \rightarrow G$, where G_1 and G are structural groups of E_1 and E correspondingly. Therefore, for the parallel transport structures we get

$$(1) F \circ \mathbf{P}_{\hat{\gamma}_1,v}^1(x) = \mathbf{P}_{\hat{\gamma},u}(x)$$

for each $x \in \hat{M}$, where $F(v) = u$, $\pi \circ F = \theta \circ \pi_1$, where \mathbf{P}^1 is for E_1 and \mathbf{P} for E . Define $F_* \langle \mathbf{P}_{\hat{\gamma}_1,v}^1 \rangle_{\alpha,\beta} := \{F \circ g : g K_{\alpha,\beta} \mathbf{P}_{\hat{\gamma}_1,v}^1\}$. Since θ and F are C_β^α differentiable mappings, then θ_* and F_* are embeddings of C_β^α differentiable spaces and group homomorphisms of C_β^l differentiable groups (see also Theorems 6 in Section 3 [15]).

2. By the conditions of this theorem N_1 is a covering of N , that is each $y \in N$ has a neighborhood V_y for which $\theta^{-1}(V_y)$ is a disjoint union of open subsets in N_1 . If an open covering \mathcal{V} of \bar{M} and a function $f \in C_\beta^\alpha(\bar{M}, N)$ are such that for each $\nu \in \mathcal{V}$ there exist $y \in N$ and V_y as above for which the embedding $f(\nu) \subset V_y$ is satisfied, then $f_1 \in C_\beta^\alpha(\bar{M}, N_1)$ exists so that $\theta \circ f_1 = f$. If $\gamma \in C_\beta^\alpha((\bar{M}, \{s_{0,q} : q = 1, \dots, k\}); (N, y_0))$, then there exists $\gamma_1 \in C_\beta^\alpha((\bar{M}, \{s_{0,q} : q = 1, \dots, k\}); (N_1, y_1))$ such that $\theta \circ \gamma_1 = \gamma$. This γ_1 evidently exists due to total disconnectedness of \bar{M} and $\gamma(\bar{M})$ and the choice axiom [3], where $\gamma(\bar{M}) \subset N$. To each parallel transport in E_1 there corresponds a parallel transport in E so that Equation (1) above is satisfied. Put $\theta_*^{-1} \langle \gamma \rangle_{\alpha,\beta} = \{\langle \gamma_1 \rangle_{\alpha,\beta} : \theta \circ \gamma_1 = \gamma\}$ and $F_*^{-1} \langle \mathbf{P}_{\hat{\gamma},u} \rangle_{\alpha,\beta} := \{\langle \mathbf{P}_{\hat{\gamma}_1,v}^1 \rangle_{\alpha,\beta} : F \circ \mathbf{P}_{\hat{\gamma}_1,v}^1 = \mathbf{P}_{\hat{\gamma},u}\}$, where $F(v) = u$.

Thus we obtain quotient mappings θ_* and F_* from closed subgroups $\theta_*^{-1}(W^MN)_{\alpha,\beta}$ and $F_*^{-1}(W^ME)_{\alpha,\beta}$ in $(W^MN_1)_{\alpha,\beta}$ and $(W^ME_1)_{\alpha,\beta}$ respectively onto $(W^MN)_{\alpha,\beta}$ and $(W^ME)_{\alpha,\beta}$ by closed subgroups $\theta_*^{-1}(e)$ and $F_*^{-1}(e)$ correspondingly.

3. We have that $g \in C_\beta^\alpha(\bar{M}, \{s_{0,q} : q = 1, \dots, k\}; \mathcal{W}, y_0)$ if and only if $f_2 \circ g \circ f_1^{-1} \in C_\beta^\alpha(\bar{M}_1, \{s_{0,q,1} : q = 1, \dots, k\}; \mathcal{W}_1, y_1)$, where $f_1(s_{0,q}) = s_{0,q,1}$ for each $q = 1, \dots, k$, $f_2(y_0 \times e) = y_1 \times e$. At the same time $\psi \in Di_\beta^\alpha(\bar{M})$ if and only if $f_1 \circ \psi \circ f_1^{-1} \in Di_\beta^\alpha(\bar{M}_1)$ (see also §3.2 [15]). Hence $(S^ME)_{\alpha,\beta}$ is isomorphic with $(S^{M_1}E_1)_{\alpha,\beta}$ and inevitably wrap groups $(W^ME)_{\alpha,\beta}$ and $(W^{M_1}E_1)_{\alpha,\beta}$ are C_β^α diffeomorphic as differentiable spaces and isomorphic as C_β^l groups.

12. Note. Let G be a topological group not necessarily associative, but alternative:

(A1) $g(gf) = (gg)f$ and $(fg)g = f(gg)$ and $g^{-1}(gf) = f$ and $(fg)g^{-1} = f$ for each $f, g \in G$

and having a conjugation operation which is a continuous automorphism of G such that

(C1) $\text{conj}(gf) = \text{conj}(f)\text{conj}(g)$ for each $g, f \in G$,

(C2) $\text{conj}(e) = e$ for the unit element e in G .

If G is of definite class of smoothness, for example, C^α_β differentiable, then conj is supposed to be of the same class. For a commutative group in particular the identity mapping as the conjugation can be taken. For $G = \mathcal{A}_r^*$ the usual conjugation $\text{conj}(z) = \tilde{z}$ can be taken for each $z \in \mathcal{A}_r^*$, where $1 \leq r \leq 3$.

Suppose that

(A2) $\hat{G} = \hat{G}_0 u_0 \oplus \hat{G}_1 u_1 \oplus \dots \oplus \hat{G}_{2^r-1} u_{2^r-1}$ such that G is a multiplicative group of a ring \hat{G} with the multiplicative group structure, where $\hat{G}_0, \dots, \hat{G}_{2^r-1}$ are pairwise isomorphic commutative associative rings and $\{u_0, \dots, u_{2^r-1}\}$ are generators of the Cayley-Dickson algebra \mathcal{A}_r over a commutative field \mathbf{K} , $1 \leq r \leq 3$ and $(y_l u_l)(y_s u_s) = (y_l y_s)(u_l u_s)$ is the natural multiplication of any pure states in G for $y_l \in G_l$. For example, $G = (\mathcal{A}_r^*)^n$ and $\hat{G} = \mathcal{A}_r^n$.

13. Lemma. *If G and K are two topological or differentiable groups twisted over $\{u_0, \dots, u_{2^r-1}\}$ satisfying conditions 12(A1, A2, C1, C2) and K is a closed normal subgroup in G , where $2 \leq r \leq 3$, then the quotient group is topological or differentiable and twisted over $\{u_0, \dots, u_{2^r-1}\}$.*

Proof. By the conditions of this lemma $\hat{G} = \hat{G}_0 u_0 \oplus \hat{G}_1 u_1 \oplus \dots \oplus \hat{G}_{2^r-1} u_{2^r-1}$, where $\hat{G}_0, \dots, \hat{G}_{2^r-1}$ are pairwise isomorphic. Then $\hat{G}/\hat{K} = (\hat{G}_0/\hat{K}_0)u_0 \oplus \dots \oplus (\hat{G}_{2^r-1}/\hat{K}_{2^r-1})u_{2^r-1}$ is also twisted. Each \hat{G}_j is associative, hence G/K is alternative, since $2 \leq r \leq 3$ and using multiplicative properties of generators of the Cayley-Dickson algebra \mathcal{A}_r . On the other hand, $\text{conj}(K) = K$, hence $\text{conj}(gK) = K\text{conj}(g) = \text{conj}(g)K \in G/K$ and $\text{conj}(ghK) = \text{conj}(gh)K = (\text{conj}(h)\text{conj}(g))K = (\text{conj}(h)K)(\text{conj}(g)K) = \text{conj}(hK)\text{conj}(gK) = \text{conj}(gKhK)$.

The subgroup K is closed in G ,

We recall the following definitions. If G_1 and G_2 are two differentiable groups, then their product is supplied with the less fine plot structure for which canonical projections $\pi_1 : G \rightarrow G_1$ and $\pi_2 : G \rightarrow G_2$ are differentiable morphisms. That is a family P of plots of G is such that $\pi_j \circ P$ is an initial family P_j of plots of G_j for $j = 1$ and $j = 2$. For differentiable groups as

usually \mathbf{P} and \mathbf{P}_j are preserved relative to the inversions and multiplications in G and G_j respectively.

If G and F are two differentiable groups with families of plots $\mathbf{P} = \mathbf{P}_G$ and \mathbf{P}_F an algebraic morphism $\theta : F \rightarrow G$ is a differentiable morphism if for each $h \in \mathbf{P}_F$ the inclusion $\theta \circ h \in \mathbf{P}_G$ follows.

For two groups G and F with an algebraic embedding $\theta : F \hookrightarrow G$ we supply F with an induced differentiable structure so that a family of plots \mathbf{P}_F of F is the less fine for which θ is a differentiable morphism.

If G is a differentiable group and F is its algebraically normal subgroup, we supply the quotient group G/F with a differentiable structure so that $\mathbf{P}_{G/F}$ is the most fine plot structure for which the quotient mapping $\theta : G \rightarrow G/F$ is a differentiable morphism.

Thus by the definition of the quotient differentiable structure G/K is the differentiable group.

14. Proposition. *Let $\eta : N_1 \rightarrow N_2$ be a $C_{\beta}^{\alpha'}$ -retraction of $C_{\beta}^{\alpha'}$ differentiable spaces, $N_2 \subset N_1$, $\eta|_{N_2} = id$, $y_0 \in N_2$, where $\alpha' \geq \alpha$, \bar{M} is an C_{β}^{α} differentiable space, $E(N_1, G, \pi, \Psi)$ and $E(N_2, G, \pi, \Psi)$ are principal $C_{\beta}^{\alpha'}$ bundles with a structure group G satisfying conditions of §§1, 2 in Section 3 [15]. Then η induces the group homomorphism η_* from $(W^M E; N_1, G, \mathbf{P})_{\alpha, \beta}$ onto $(W^M E; N_2, G, \mathbf{P})_{\alpha, \beta}$.*

Proof. Due to Proposition 6(1) the wrap group $(W^M E; N_1, G, \mathbf{P})_{\alpha, \beta}$ is the principal G^k bundle over $(W^M N_1)_{\alpha, \beta}$. We extend η to $\vartheta : E(N_1, G, \pi, \Psi) \rightarrow E(N_2, G, \pi, \Psi)$ such that $\pi_2 \circ \vartheta = \eta \circ \pi_1$ and $pr_2 \circ \vartheta = id : G \rightarrow G$, where $pr_2 : E_y \rightarrow G$ is the projection, $y \in N_1$, $\eta(N_1) = N_2$. If $f \in C_{\beta}^{\alpha}(\bar{M}, N_1)$, then $\eta \circ f := \eta(f(*)) \in C_{\beta}^{\alpha}(\bar{M}, N_2)$. If $f(s_{0,q}) = y_0$, then $\eta(f(s_{0,q})) = y_0$, since $y_0 \in N_2$. From the inclusion $N_2 \subset N_1$ we deduce that $C_{\beta}^{\alpha}(\bar{M}, N_2) \subset C_{\beta}^{\alpha}(\bar{M}, N_1)$. The parallel transport structure \mathbf{P} is given over the same differentiable space \bar{M} .

Put $\eta_*(\langle \mathbf{P}_{\hat{\gamma}, u} \rangle_{\alpha, \beta}) = \langle \mathbf{P}_{\eta \circ \hat{\gamma}, u} \rangle_{\alpha, \beta}$, where $\hat{\gamma} : \hat{M} \rightarrow N_1$. In accordance with Theorems 2.3 and 2.6 in Section 3 [15] $\eta_*(\langle \mathbf{P}_{\hat{\gamma}_1, u} \vee \mathbf{P}_{\hat{\gamma}_2, u} \rangle_{\alpha, \beta}) = \eta^*(\langle \mathbf{P}_{\hat{\gamma}_1, u} \rangle_{\alpha, \beta}) \eta_*(\langle \mathbf{P}_{\hat{\gamma}_2, u} \rangle_{\alpha, \beta})$, and we can put $\eta_*(q^{-1}) = [\eta_*(q)]^{-1}$, consequently, η_* is the group homomorphism. Moreover, for each $g \in (W^M E; N_2, G, \mathbf{P})_{\alpha, \beta}$ there exists $q \in (W^M E; N_1, G, \mathbf{P})_{\alpha, \beta}$ such that $\eta_*(q) = g$, since $\gamma : \bar{M} \rightarrow N_2$ and $N_2 \subset N_1$ imply $\gamma : \bar{M} \rightarrow N_1$. On the other hand, the structure group G is the same, hence η_* is the epimorphism.

15. Definition. Suppose that G is a topological group satisfying Conditions 12(A1, A2, C1, C2) such that G is a multiplicative group of the ring

\hat{G} , where $1 \leq r \leq 2$. We define a smashed product G^s such that it is a multiplicative group of the ring $\hat{G}^s := \hat{G} \otimes_l \hat{G}$, where $l = u_{2^r}$ denotes the doubling generator, a multiplication in $\hat{G} \otimes_l \hat{G}$ is given by the formula:

(1) $(a + bl)(c + vl) = (ac - v^*b) + (va + bc^*)l$ for each $a, b, c, v \in \hat{G}$, where $v^* = \text{conj}(v)$.

In this relation it is worth to mention that \mathcal{A}_r with $r \leq 3$ is the division algebra. For matrices with entries in \mathcal{A}_r the Gauss' algorithm is valid, so matrices have ranks by rows and columns which coincide and so a dimension over \mathcal{A}_r is defined [1].

A smashed product $M_1 \otimes_l M_2$ of C_β^α differentiable manifolds M_1, M_2 over \mathcal{A}_r with $\dim_{\mathcal{A}_r}(M_1) = \dim_{\mathcal{A}_r}(M_2)$ is defined to be an \mathcal{A}_{r+1} differentiable manifold with local coordinates $z = (x, yl)$ of class C_β^α , where x in M_1 and y in M_2 are local coordinates.

Its existence and detailed description are demonstrated below.

16. Proposition. *The ring \hat{G}^s from §15 has a multiplicative group G^s containing all $a + bl \neq 0$ with $a, b \in \hat{G}$. If \hat{G} is a topological or C_β^α differentiable ring over \mathcal{A}_r , then \hat{G}^s is a topological or C_β^α differentiable over \mathcal{A}_{r+1} ring.*

Proof. If $1 \leq r \leq 2$ then a group G is associative, since the generators $\{u_0, \dots, u_{2^r-1}\}$ form the associative group, when $r \leq 2$. An element $a + bl \in \hat{G}^s$ is non-zero if and only if $(a + bl)(a + bl)^* = aa^* + bb^* \neq 0$ due to 12(A1, A2, C1, C2) and 15(1). For $a + bl \neq 0$ we put $u = (a^* - lb^*)/(aa^* + bb^*)$, where $aa^* + bb^* \in G_0$, hence $u(a + bl) = (a + bl)u = 1 \in G_0$, since G_j is commutative for each $j = 0, \dots, 2^r - 1$, where G_j denotes the multiplicative group of the ring \hat{G}_j . The family of generators $\{u_0, \dots, u_{2^{r+1}-1}\}$ for $r \leq 2$ forms the alternative group, hence $\hat{G}^s = \hat{G}_0 u_0 \oplus \dots \oplus \hat{G}_{2^{r+1}-1} u_{2^{r+1}-1}$ is alternative, where \hat{G}_j are isomorphic with \hat{G}_0 for each j .

If an operation of the addition in \hat{G} is continuous, then evidently $(a + bl) + (c + ql) = (a + c) + (b + q)l$ is continuous. If an operation of the multiplication in \hat{G} is continuous, then Formula 15(1) shows that the multiplication in \hat{G}^s is continuous as well.

We have the decomposition $\mathcal{A}_{r+1} = \mathcal{A}_r \oplus \mathcal{A}_r l$. If \hat{G} is C_β^α differentiable, then from the definition of plots it follows, that \hat{G}^s is C_β^α differentiable over \mathcal{A}_{r+1} (see also in details 17(1 – 5)).

17. Theorem. *Let \bar{M}_1, \bar{M}_2 and N_1, N_2 be C_β^α differentiable manifolds over \mathcal{A}_r with $1 \leq r \leq 2$, and let G be a group satisfying Conditions*

12(A1, A2, C1, C2). Suppose also that $\bar{M}_1 \otimes_l \bar{M}_2$, $N_1 \otimes_l N_2$ are smashed products of differentiable manifolds and G^s is a smashed product group (see Proposition 16), where $\dim_{\mathcal{A}_r}(\bar{M}_1) = \dim_{\mathcal{A}_r}(\bar{M}_2)$, $\dim_{\mathcal{A}_r}(N_1) = \dim_{\mathcal{A}_r}(N_2)$. Then the wrap group

$(W^{M_1 \otimes_l M_2; \{s_{0,j,1} \otimes_l s_{0,v,2}; j=1, \dots, k_1; v=1, \dots, k_2\}} E; N_1 \otimes_l N_2, G^s, \mathbf{P}^s)_{\alpha, \beta}$ is twisted over $\{u_0, \dots, u_{2^{r+1}-1}\}$ and is isomorphic with the smashed product $(W^{M_2; \{s_{0,v,2}; v=1, \dots, k_2\}} E; N_1, (W^{M_1; \{s_{0,j,1}; j=1, \dots, k_1\}} E; N_1, G, \mathbf{P}_1)_{\alpha, \beta}, \mathbf{P}_2)_{\alpha, \beta} \otimes_l (W^{M_2; \{s_{0,v,2}; v=1, \dots, k_2\}} E; N_2, (W^{M_1; \{s_{0,j,1}; j=1, \dots, k_1\}} E; N_2, G, \mathbf{P}_1)_{\alpha, \beta}, \mathbf{P}_2)_{\alpha, \beta}$ of twice iterated wrap groups twisted over $\{u_0, \dots, u_{2^r-1}\}$.

Proof. Let M_b and N_b be C_β^α differentiable manifolds over \mathcal{A}_r with $1 \leq r \leq 2$, $b = 1, 2$ and let G be a group satisfying Conditions 12(A1, A2, C1, C2) such that $E(N_b, G, \pi, \Psi)$ is a principal G -bundle. We consider the smashed products $\bar{M}_1 \otimes_l \bar{M}_2$, $N_1 \otimes_l N_2$ of C_β^α differentiable manifolds and the smashed product group G^s (see Proposition 16). For $At(\bar{M}_b) = \{(U_{j,b}, \phi_{j,b}) : j\}$ an atlas of \bar{M}_b its connecting mappings $\phi_{j,b} \circ \phi_{k,b}^{-1}$ are C_β^α functions over \mathcal{A}_r for $U_{j,b} \cap U_{k,b} \neq \emptyset$, where $\phi_{j,b} : U_{j,b} \rightarrow \mathcal{A}_r$ are homeomorphisms of $U_{j,b}$ onto $\phi_{j,b}(U_{j,b})$. Then $\bar{M}_1 \otimes_l \bar{M}_2$ consists of all points (x, yl) with $x \in \bar{M}_1$ and $y \in \bar{M}_2$, with the atlas $At(\bar{M}_1 \otimes_l \bar{M}_2) = \{(U_{j,1} \otimes_l U_{q,2}, \phi_{j,1} \otimes_l \phi_{q,2}) : j, q\}$ such that $\phi_{j,1} \otimes_l \phi_{q,2} : U_{j,1} \otimes_l U_{q,2} \rightarrow \mathcal{A}_{r+1}^m$, where m is a dimension of \bar{M}_1 over \mathcal{A}_r . Express for $z = x + yl \in \mathcal{A}_{r+1}$ with $x, y \in \mathcal{A}_r$ numbers x, y in the z representation. Then we denote by $\theta_{j,q}$ mappings corresponding to $\phi_{j,1} \otimes_l \phi_{q,2}$ in the z representation. Thus the transition mappings $\theta_{j,q} \circ \theta_{k,n}^{-1}$ are C_β^α differentiable over \mathbf{K} , when $(U_{j,1} \otimes_l U_{q,2}) \cap (U_{k,1} \otimes_l U_{n,2}) \neq \emptyset$. Therefore, $\bar{M}_1 \otimes_l \bar{M}_2$ and $N_1 \otimes_l N_2$ are C_β^α differentiable manifolds over \mathcal{A}_{r+1} .

Each C_β^α function $f(x, y) = f_1(x, y) + f_2(x, y)l$ by $x \in U$ and $y \in V$ is of class C_β^α by variables over \mathbf{K} and takes values in $X_N \oplus X_N l$ over \mathcal{A}_{r+1} , where U and V are open in X_M , $f_b(x, y)$ is a C_β^α function with values in X_N over \mathcal{A}_r , $b = 1, 2$. If $z \in \mathcal{A}_{r+1}$, then

- (1) $z = v_0 u_0 + \dots + v_{2^{r+1}-1} u_{2^{r+1}-1}$, where $v_j \in \mathbf{K}$ for each $j = 0, \dots, 2^{r+1}-1$,
- (2) $v_0 = (z + (2^{r+1} - 2)^{-1} \{-z + \sum_{j=1}^{2^{r+1}-1} u_j(zu_j^*)\})/2$,
- (3) $v_s = (u_s(2^{r+1} - 2)^{-1} \{-z + \sum_{j=1}^{2^{r+1}-1} u_j(zu_j^*)\} - zu_s)/2$ for each $s = 1, \dots, 2^{r+1}-1$, where $z^* = \tilde{z}$ denotes the conjugated Cayley-Dickson number z . At the same time we have for $z = x + yl$ with $x, y \in \mathcal{A}_r$, that
- (4) $x = v_0 u_0 + \dots + v_{2^r-1} u_{2^r-1}$ and
- (5) $y = (v_{2^r} u_{2^r} + \dots + v_{2^{r+1}-1} u_{2^{r+1}-1})l^*$,

where $l = u_{2^r}$ denotes the doubling generator. Therefore, using Formulas

(1 – 5) we get, that $f(x, y)$ is C_β^α differentiable over \mathbf{K} .

Then $E(N_1 \otimes_l N_2, G^s, \pi^s, \Psi^s)$ is naturally isomorphic with $E(N_1, G, \pi_1, \Psi_1) \otimes_l E(N_2, G, \pi_2, \Psi_2)$, where $\pi^s = \pi_1 \otimes \pi_2 l : E(N_1 \otimes_l N_2, G^s, \pi^s, \Psi^s) \rightarrow N_1 \otimes_l N_2$ is the natural projection.

If $\gamma : \bar{M}_1 \otimes_l \bar{M}_2 \rightarrow N_1 \otimes_l N_2$ is an C_β^α mapping, then $\gamma(z) = \gamma_1(x, y) \times \gamma_2(x, y)l$, where $x \in \bar{M}_1$ and $y \in \bar{M}_2$, $z = (x, yl) \in \bar{M}_1 \otimes_l \bar{M}_2$, $\gamma_b : \bar{M}_1 \otimes_l \bar{M}_2 \rightarrow N_b$. We can write $\gamma_b(x, y)$ as $(\gamma_{b,1}(x))(y)$ a family of functions by x and a parameter y or as $(\gamma_{b,2}(y))(x)$ a family of functions by y with a parameter x . If $\eta_{b,a} : \bar{M}_a \rightarrow N_b$, then $\mathbf{P}_{\hat{\eta}_{b,a}, u_b, a}$ denotes the parallel transport structure on \bar{M}_a over $E(N_b, G, \pi_b, \Psi_b)$.

Then we obtain

$$\mathbf{P}_{\hat{\gamma}, u}^s(z) = [\mathbf{P}_{\hat{\gamma}_{1,1}, u_1; 1}(x)][\mathbf{P}_{\hat{\gamma}_{1,2}, u_1; 2}(y)] \otimes_l [\mathbf{P}_{\hat{\gamma}_{2,1}, u_2; 2}(x)][\mathbf{P}_{\hat{\gamma}_{2,2}, u_2; 2}(y)] \in E_{y_0}(N_1 \otimes_l N_2, G^s, \pi^s, \Psi^s)$$

is the parallel transport structure in $\bar{M}_1 \otimes_l \bar{M}_2$ induced by that of in \bar{M}_1 and \bar{M}_2 , where $u \in E_{y_0}(N_1 \otimes_l N_2, G^s, \pi^s, \Psi^s)$, $u = u_1 \otimes_l u_2$, $u_b \in E_{y_{0,b}}(N_b, G, \pi_b, \Psi_b)$, $y_{0,b} \in N_b$ is a marked point, $b = 1, 2$, $y_0 = y_{0,1} \otimes_l y_{0,2}$. Thus \mathbf{P}^s is G^s equivariant. Therefore, the formula $\langle \mathbf{P}_{\hat{\gamma}, u}^s \rangle_{\alpha, \beta} = \langle \mathbf{P}_{\hat{\gamma}_1, u_1}^s \rangle_{\alpha, \beta} \otimes_l \langle \mathbf{P}_{\hat{\gamma}_2, u_2}^s \rangle_{\alpha, \beta} = \langle [\mathbf{P}_{\hat{\gamma}_{1,1}, u_1; 1}(x)][\mathbf{P}_{\hat{\gamma}_{1,2}, u_1; 2}(y)] \rangle_{\alpha, \beta} \otimes_l \langle [\mathbf{P}_{\hat{\gamma}_{2,1}, u_2; 2}(x)][\mathbf{P}_{\hat{\gamma}_{2,2}, u_2; 2}(y)] \rangle_{\alpha, \beta}$, where $\mathbf{P}_{\hat{\gamma}_b, u_b}$ produces the parallel transport structure in $\bar{M}_1 \otimes_l \bar{M}_2$ over $E(N_b, G, \pi_b, \Psi_b)$, $b = 1, 2$.

Hence the group $(W^{M_1 \otimes_l M_2; \{s_{0,j,1} \otimes_l s_{0,v,2}; j=1, \dots, k_1; v=1, \dots, k_2\}} E; N_1 \otimes_l N_2, G^s, \mathbf{P}^s)_{\alpha, \beta}$ is isomorphic with the smashed product

$$(W^{M_2; \{s_{0,v,2}; v=1, \dots, k_2\}} E; N_1, (W^{M_1; \{s_{0,j,1}; j=1, \dots, k_1\}} E; N_1, G, \mathbf{P}_1)_{\alpha, \beta}, \mathbf{P}_2)_{\alpha, \beta} \otimes_l (W^{M_2; \{s_{0,v,2}; v=1, \dots, k_2\}} E; N_2, (W^{M_1; \{s_{0,j,1}; j=1, \dots, k_1\}} E; N_2, G, \mathbf{P}_1)_{\alpha, \beta}, \mathbf{P}_2)_{\alpha, \beta}$$

of iterated wrap groups.

18. Theorem. *A homomorphism of iterated wrap groups $\theta : (W^M E)_{a; \infty, \beta} \otimes (W^M E)_{b; \infty, \beta} \rightarrow (W^M E)_{a+b; \infty, \beta}$ exists for each $a, b \in \mathbf{N}$, where G is a C_β^∞ group, $E(N, G, \pi, \Psi)$ is a principal C_β^∞ bundle with a structure group G . Moreover, if G is either associative or alternative, then the homomorphism θ is either associative or alternative correspondingly.*

Proof. We consider iterated wrap groups $(W^M E)_{a; \infty, \beta}$ as in §3, $a \in \mathbf{N}$. If $\gamma_a : \bar{M}^a \rightarrow N$, $\gamma_b : \bar{M}^b \rightarrow N$ are C_β^∞ mappings such that $\gamma_b(s_{0,j_1} \times \dots \times s_{0,j_b}) = y_0$ for each $j_l = 1, \dots, k$ and $l = 1, \dots, b$, then $\gamma := \gamma_a \times \gamma_b : \bar{M}^a \times \bar{M}^b \rightarrow N \times N = N^2$, where $\bar{M}^a \times \bar{M}^b = \bar{M}^{a+b}$, $s_{0,j}$ are marked points in M with $j = 1, \dots, k$ and y_0 is a marked point in N , $C_\beta^\infty = \bigcap_{\alpha \in \mathbf{N}} C_\beta^\alpha$. This gives the iterated parallel transport structure $\mathbf{P}_{\hat{\gamma}, u; a+b}(x) := \mathbf{P}_{\hat{\gamma}_a, u_a; a}(x_a) \otimes \mathbf{P}_{\hat{\gamma}_b, u_b; b}(x_b)$ on \bar{M}^{a+b} over $E(N^2, G^2, \pi, \Psi)$, where $u_b \in E_{y_0}(N, G, \pi, \Psi)$, $u = u_a \times u_b \in$

$E_{y_0 \times y_0}(N^2, G^2, \pi, \Psi)$.

The bunch $\bar{M}^b \vee \bar{M}^b$ is taken by points s_{j_1, \dots, j_b} in \bar{M}^b , where $s_{j_1, \dots, j_b} := s_{0, j_1} \times \dots \times s_{0, j_b}$ with $j_1, \dots, j_b \in \{1, \dots, k\}$; $s_{0, j}$ are marked points in \bar{M} with $j = 1, \dots, k$. Then the differentiable space $(\bar{M}^a \vee \bar{M}^a) \times (\bar{M}^b \vee \bar{M}^b) \setminus \{s_{j_1, \dots, j_{a+b}} : j_l = 1, \dots, k; l = 1, \dots, a+b\}$ is C_β^α homeomorphic with $\bar{M}^{a+b} \vee \bar{M}^{a+b} \setminus \{s_{j_1, \dots, j_{a+b}} : j_l = 1, \dots, k; l = 1, \dots, a+b\}$, since $s_{j_1, \dots, j_a} \times s_{j_{a+1}, \dots, j_{a+b}} = s_{j_1, \dots, j_{a+b}}$ for each j_1, \dots, j_{a+b} . We have also the embedding $Di_\beta^\infty(\bar{M}^a) \times Di_\beta^\infty(\bar{M}^b) \hookrightarrow Di_\beta^\infty(\bar{M}^{a+b})$ for each $a, b \in \mathbf{N}$ (see also §3.2 [15]). If $f_a \in Di_\beta^\infty(\bar{M}^a)$ having a restriction $f_a|_{K_a} = id$, then $f_a \times f_b \in Di_\beta^\infty(\bar{M}^{a+b})$ and $f_a \times f_b|_{K_a \times K_b} = id$ for $K_a \subset M^a$. Put

$$\theta(< \mathbf{P}_{\hat{\gamma}_a, u_a; a} >_{K, \infty, \beta; a}, < \mathbf{P}_{\hat{\gamma}_b, u_b; b} >_{K, \infty, \beta; b}) =$$

$$<< \mathbf{P}_{\hat{\gamma}_a, u_a; a} >_{K, \infty, \beta; a} \otimes < \mathbf{P}_{\hat{\gamma}_b, u_b; b} >_{K, \infty, \beta; b} >_{K, \infty, \beta; a+b},$$

so it is the group homomorphism, where the detailed notation $< * >_{K, \alpha, \beta; a}$ means the equivalence class over the differentiable space \bar{M}^a instead of \bar{M} , $a \in \mathbf{N}$.

Therefore, $< \mathbf{P}_{\hat{\gamma} \vee \hat{\eta}, u; a+b} >_{K, \infty, \beta; a+b} :=$

$$\begin{aligned} &<< \mathbf{P}_{\hat{\gamma}_a \vee \hat{\eta}_a, u_a; a} >_{K, \infty, \beta; a} \otimes < \mathbf{P}_{\hat{\gamma}_b \vee \hat{\eta}_b, u_b; b} >_{K, \infty, \beta; b} >_{K, \infty, \beta; a+b} \\ &= < (< \mathbf{P}_{\hat{\gamma}_a, u_a; a} >_{K, \infty, \beta; a} < \mathbf{P}_{\hat{\eta}_a, u_a; a} >_{K, \infty, \beta; a}) \otimes (< \mathbf{P}_{\hat{\gamma}_b, u_b; b} >_{K, \infty, \beta; b} < \mathbf{P}_{\hat{\eta}_b, u_b; b} >_{K, \infty, \beta; b}) >_{K, \infty, \beta; a+b} \\ &= < (< \mathbf{P}_{\hat{\gamma}_a, u_a; a} >_{K, \infty, \beta; a} \otimes < \mathbf{P}_{\hat{\gamma}_b, u_b; b} >_{K, \infty, \beta; b}) (< \mathbf{P}_{\hat{\eta}_a, u_a; a} >_{K, \infty, \beta; a} \otimes < \mathbf{P}_{\hat{\eta}_b, u_b; b} >_{K, \infty, \beta; b}) >_{K, \infty, \beta; a+b} \\ &= << \mathbf{P}_{\hat{\gamma}_a, u_a; a} >_{K, \infty, \beta; a} \otimes < \mathbf{P}_{\hat{\gamma}_b, u_b; b} >_{K, \infty, \beta; b} >_{K, \infty, \beta; a+b} << \mathbf{P}_{\hat{\eta}_a, u_a; a} >_{K, \infty, \beta; a} \\ &\otimes < \mathbf{P}_{\hat{\eta}_b, u_b; b} >_{K, \infty, \beta; b} >_{K, \infty, \beta; a+b} \\ &= \theta(< \mathbf{P}_{\hat{\gamma}_a, u_a; a} >_{K, \infty, \beta; a}, < \mathbf{P}_{\hat{\gamma}_b, u_b; b} >_{K, \infty, \beta; b}) \theta(< \mathbf{P}_{\hat{\eta}_a, u_a; a} >_{K, \infty, \beta; a}, < \mathbf{P}_{\hat{\eta}_b, u_b; b} >_{K, \infty, \beta; b}). \end{aligned}$$

Thus θ is the group homomorphism.

The mapping $C_\beta^\infty(\bar{M}^a, N) \times C_\beta^\infty(\bar{M}^b, N) \ni (\gamma_a \times \gamma_b) \mapsto (\gamma_a, \gamma_b) \in C_\beta^\infty(\bar{M}^{a+b}, N^2)$ is of C_β^∞ class. The multiplication in the group G^v is defined by the formula: $(a_1, \dots, a_v) \times (b_1, \dots, b_v) = (a_1 b_1, \dots, a_v b_v)$, where G^v is the v times direct product of G , $a_1, \dots, a_v, b_1, \dots, b_v \in G$. Therefore, the multiplication in G^v is C_β^∞ smooth for each $v \in \mathbf{N}$, since it is such in G .

The iterated wrap group $(W^M E)_{l, \alpha, \beta}$ for the bundle E is the principal G^{kl} bundle over the iterated commutative wrap group $(W^M N)_{l, \alpha, \beta}$ for the manifold N , since the number of marked points in M^l is kl , where E is the principal G bundle on the manifold N , $l \in \mathbf{N}$. Thus the iterated wrap group is associative or alternative if such is G . In view of Proposition 6 and §3 the

homomorphism θ is of C_β^∞ class. From the wrap monoids it has the natural C_β^∞ extension on wrap groups.

If G is associative, then

$$\begin{aligned} & \langle \mathbf{P}_{\hat{\gamma}, u; a+b+v} \rangle_{K, \infty, \beta; a+b+v} = \langle \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{K, \infty, \beta; b} \rangle_{K, \infty, \beta; a+b} \otimes \langle \mathbf{P}_{\hat{\gamma}_v, u_v; v} \rangle_{K, \infty, \beta; v} \rangle_{K, \infty, \beta; a+b+v} \\ & = \langle \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{K, \infty, \beta; b} \otimes \langle \mathbf{P}_{\hat{\gamma}_v, u_v; v} \rangle_{K, \infty, \beta; v} \rangle_{K, \infty, \beta; a+b+v} = \\ & \theta(\theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a}, \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{K, \infty, \beta; b}), \langle \mathbf{P}_{\hat{\gamma}_v, u_v; v} \rangle_{K, \infty, \beta; v}) \\ & \theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a}, \theta(\langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{K, \infty, \beta; b}, \langle \mathbf{P}_{\hat{\gamma}_v, u_v; v} \rangle_{K, \infty, \beta; v})), \end{aligned}$$

consequently, θ is the associative homomorphism.

If G is alternative, then

$$\begin{aligned} & \langle \mathbf{P}_{\hat{\gamma}, u; a+a+b} \rangle_{K, \infty, \beta; a+a+b} = \langle \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a} \rangle_{K, \infty, \beta; a+a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{K, \infty, \beta; b} \rangle_{K, \infty, \beta; a+a+b} \\ & = \langle \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{K, \infty, \beta; b} \rangle_{K, \infty, \beta; a+a+b} = \\ & \theta(\theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a}, \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a}), \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{K, \infty, \beta; b}) = \\ & \theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a}, \theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{K, \infty, \beta; a}, \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{K, \infty, \beta; b})). \end{aligned}$$

Thus the homomorphism θ is alternative from the left, analogously it is alternative from the right.

19. Remark. Wrap groups were defined and studied above for fiber bundles over a field \mathbf{K} and algebras \mathcal{A}_r with $1 \leq r \leq 3$ with a topological or differentiable structure group G which may be of Lie type as well. In particular this encompasses the case of multiplicative groups G of commutative algebras such as H_n of diagonal matrices with entries in \mathbf{K} , particularly, H_4 of quadra numbers.

It is interesting to mention that using an extension $\mathbf{F} \subset \mathbf{K}$ of a field \mathbf{F} up to a field \mathbf{K} of a positive characteristic p it is possible to construct a division algebra of a dimension $m = n^2$ greater than eight over \mathbf{F} , where n is a natural number [1]. The method of construction uses irreducible polynomials over \mathbf{F} . This \mathbf{K} may be a finite field F_{p^k} and a locally compact field of fractions $F_{p^k}(\theta)$ over F_{p^k} with an indeterminate θ , where $k \in \mathbf{N}$, that was outlined by Dickson. If such algebra is constructed over F_{p^k} , then it exists over $F_{p^k}(\theta)$ also. Indeed, each irreducible polynomial $g_l(x) := x^l + c_{l-1}x^{l-1} + \dots + c_1x + c_0$ with expansion coefficients $c_0, \dots, c_{l-1} \in F_{p^k}$, $c_0 \neq 0$, is also irreducible over $F_{p^k}(\theta)$, since $|c_j| = 1$ for $c_j \neq 0$ and $|c_j| = 0$ for $c_j = 0$, where $|\cdot|$ is the multiplicative norm in $F_{p^k}(\theta)$ with $0 < |\theta| < 1$. Each $x \in F_{p^k}(\theta)$ has the form $x = \theta^N \sum_{j=0}^{\infty} \beta_j \theta^j$, where $\beta_j = \beta_j(x) \in F_{p^k}$ for each j , $\beta_0 \neq 0$, $N \in \mathbf{Z}$, $|x| = |\theta|^N$. Therefore, $|g_l(x)| = |\theta|^{Nl}$ for $N < 0$, since $|a+b| \leq \max(|a|, |b|)$ for all $a, b \in F_{p^k}(\theta)$. For $N > 0$ we get $|g_l(x) - c_0| = |\theta|^{Nk_0}$, where $k_0 =$

$\min\{k : c_k \neq 0, c_{k-1} = 0, \dots, c_1 = 0\}$, consequently, $|g_l(x)| = |c_0| = 1$ and such x can not be a zero of $g_l(x)$. Thus if the polynomial $g_l(x)$ has a zero $z \in F_{p^k}(\theta)$, then $|z| = 1$, since $c_0 \neq 0$. Consider terms in an expansion of $g_l(z)$ by degrees of θ . For θ^0 the corresponding coefficient $\beta_0 = \beta_0(g_l(z))$ should be equal to zero, hence a zero $x_0 \in F_{p^k}$ would exist. Thus the polynomial $g_l(x)$ is irreducible over $F_{p^k}(\theta)$ also.

Suppose now that \mathbf{R} is a ring having two subgroups. One of them G_1 is commutative related with the addition, $G_1 = (\mathbf{R}, +)$. Another is multiplicative $G_2 = (\mathbf{R} \setminus \{0\}, \times)$. Particularly, \mathbf{R} may be an algebra \mathbf{A} over the field \mathbf{K} . We consider the cases of commutative, associative and as well as non-associative rings and algebras with associative addition in G_1 and alternative multiplication in G_2 . Suppose that a fiber bundle is given with the structure ring \mathbf{R} or the structure algebra \mathbf{A} instead of a group. We suppose also that with G_1 and G_2 parallel transport structures ${}_1\mathbf{P}$ and ${}_2\mathbf{P}$ are related. So we shall say that there is the parallel transport structure \mathbf{P} on the principal fiber bundle $E(N, \mathbf{R}, \pi, \Psi)$ or $E(N, \mathbf{A}, \pi, \Psi)$.

20. Theorem. *Wrap groups $(W^{M, \{s_0, q: q=1, \dots, k\}} E; N, \mathbf{R}, \mathbf{P})_{\alpha, \beta}$ or $(W^{M, \{s_0, q: q=1, \dots, k\}} E; N, \mathbf{A}, \mathbf{P})_{\alpha, \beta}$ exist with two C_β^l group's operations and they are the principal fiber bundles over the commutative group $(W^M N)_{\alpha, \beta}$ with the structure ring \mathbf{R}^k or the structure algebra \mathbf{A}^k respectively, where $l = \alpha' - \alpha$ for $\alpha \leq \alpha' < \infty$ or $l = \infty$ for $\alpha' = \infty$ (see also Remark 19 above).*

Proof. Earlier wrap groups $(W^{M, \{s_0, q: q=1, \dots, k\}} E; N, G_j, {}_j\mathbf{P})_{\alpha, \beta}$ for $j = 1, 2$ for the principal fiber bundles $E(N, G_j, \pi_j, \Psi_j)$ having C_β^l group's operations were constructed in accordance with Theorem 6 [15]. We consider $\theta_j : G_j \hookrightarrow \mathbf{R}$ or $\theta_j : G_j \hookrightarrow \mathbf{A}$ group embedding of the $C_\beta^{\alpha'}$ class of differentiability, $j = 1, 2$. In view of Proposition 6 above they are the principal fiber bundles over $(W^M N)_{\alpha, \beta}$ with the structure groups G_j^k . At the same time the principal fiber bundle $E(N, \mathbf{R}, \pi, \Psi)$ or $E(N, \mathbf{A}, \pi, \Psi)$ is isomorphic with $E((E(N, G_2, \pi_2, \Psi_2)), G_1, \pi_1, \Psi_1)/\xi$, where the equivalence relation ξ is induced by the equality $\theta_1(x) = \theta_2(y)$ in \mathbf{R} or \mathbf{A} respectively of the corresponding elements $x \in G_1$ and $y \in G_2$. Then we put

$$(W^{M, \{s_0, q: q=1, \dots, k\}} E; N, \mathbf{T}, \mathbf{P})_{\alpha, \beta} := (W^{M, \{s_0, q: q=1, \dots, k\}} E((E(N, G_2, \pi_2, \Psi_2)), G_1, \pi_1, \Psi_1); \mathbf{P})_{\alpha, \beta}/\xi,$$

where $\mathbf{T} = \mathbf{R}$ or $\mathbf{T} = \mathbf{A}$ respectively. In addition we have put

$$(W^{M, \{s_0, q: q=1, \dots, k\}} E; K, G_j, {}_j\mathbf{P})_{\alpha, \beta} =: (W^{M, \{s_0, q: q=1, \dots, k\}} E(K, G_j, \pi_j, \Psi_j); \mathbf{P})_{\alpha, \beta},$$

consequently, $(W^{M, \{s_0, q: q=1, \dots, k\}} E; N, \mathbf{T}, \mathbf{P})_{\alpha, \beta}$ is supplied with two C_β^l group

operations corresponding to $(W^{M, \{s_{0,q}: q=1, \dots, k\}} E; N, G_{j, j} \mathbf{P})_{\alpha, \beta}$ for $j = 1, 2$ and there exists the principal fiber bundle

$$\begin{aligned} & (W^{M, \{s_{0,q}: q=1, \dots, k\}} E((E(N, G_2, \pi_2, \Psi_2)), G_1, \pi_1, \Psi_1); \mathbf{P})_{\alpha, \beta} \\ & \rightarrow (W^{M, \{s_{0,q}: q=1, \dots, k\}} E; N, G_2, {}_2\mathbf{P})_{\alpha, \beta} \end{aligned}$$

with the structure group G_1^k . Using the equivalence relation ξ inevitably infers that there exists the principal fiber bundle

$$(W^{M, \{s_{0,q}: q=1, \dots, k\}} E; N, \mathbf{T}, \mathbf{P})_{\alpha, \beta} \rightarrow (W^M N)_{\alpha, \beta}$$

with the structure ring \mathbf{R}^k or the structure algebra \mathbf{A}^k correspondingly.

21. Remark. Wrap groups and semigroups can be generalized for an infinite discrete closed subset $\{s_{0,q} : q \in \lambda\}$ of marked points in \bar{M} , when \bar{M} is not compact, where λ is a set, $\text{card}(\lambda) \geq \aleph_0$.

Apart from the differentiable spaces M, N, E considered above over the infinite non-discrete field \mathbf{K} analogous wrap groups also exist when M, N and E are over a finite field F_{p^k} or an algebra \mathcal{A}_r over F_{p^k} , $1 \leq r \leq 3$, but such wrap groups become already discrete. Particularly, if M, N and E are finite, then wrap groups are finite (see also References [25,26] in [15]). Therefore, we have considered above topological infinite groups, when \mathbf{K} is infinite and non-discrete.

References

- [1] L.E. Dickson. "The collected mathematical papers". Volumes 1-5. (New York: Chelsea Publishing Co., 1975).
- [2] G. Emch. "Mèchanique quantique quaternionnienne et relativité restreinte". Helv. Phys. Acta **36** (1963), 739-788.
- [3] R. Engelking. "General topology" (Moscow: Mir, 1986).
- [4] P. Gajer. "Higher holonomies, geometric loop groups and smooth Deligne cohomology". In: "Advances in geometry". Progr. in Math. **172**, 195-235 (Boston: Birkhäuser, 1999).
- [5] F. Gürsey, C.-H. Tze. "On the role of division, Jordan and related algebras in particle physics" (Singapore: World Scientific Publ. Co., 1996).
- [6] F.R. Harvey. "Spinors and calibrations". Perspectives in Mathem. **9** (Boston: Academic Press, 1990).

- [7] C.J. Isham. "Topological and global aspects of quantum theory". In: "Relativity, groups and topology.II" 1059-1290, (Les Hauches, 1983). Editors: R. Stora, B.S. De Witt (Amsterdam: Elsevier Sci. Publ., 1984).
- [8] H.B. Lawson, M.-L. Michelson. "Spin geometry" (Princeton: Princ. Univ. Press, 1989).
- [9] S.V. Ludkovsky. "Quasi-invariant measures on loop groups of Riemann manifolds". Dokl. Akad. Nauk **370: 3** (2000), 306-308.
- [10] S.V. Ludkovsky. "Poisson measures for topological groups and their representations". Southeast Asian Bulletin of Mathematics. **25** (2002), 653-680.
- [11] S.V. Ludkovsky. "Differentiability of functions: approximate, global and differentiability along curves over non-archimedean fields". J. Mathem. Sci. **157: 2** (2009), 311-366
(previous variant: Los Alamos Nat. Lab. math.CA/0608724).
- [12] S.V. Ludkovsky. "Stochastic processes on geometric loop groups, diffeomorphism groups of connected manifolds, associated unitary representations". J. Mathem. Sci. **141: 3** (2007), 1331-1384 (previous version: Los Alam. Nat. Lab. math.AG/0407439, July 2004).
- [13] S.V. Ludkovsky. "Geometric loop groups and diffeomorphism groups of manifolds, stochastic processes on them, associated unitary representations". In the book: "Focus on Groups Theory Research" (New York: Nova Science Publishers, Inc., 2006) pages 59-136.
- [14] S.V. Ludkovsky. "Generalized geometric loop groups of complex manifolds, Gaussian quasi-invariant measures on them and their representations". J. Mathem. Sci. **122: 1** (2004), 2984-3011 (earlier version: Los Alam. Nat. Lab. math.RT/9910086, October 1999).
- [15] S. V. Ludkovsky "Wrap groups of non-archimedean fiber bundles". Los Alamos Nat. Lab. Archives math.GR/1201.4905.
- [16] M.B. Mensky. "The paths group. Measurement. Fields. Particles" (Moscow: Nauka, 1983).

- [17] J. Milnor. "Morse theory" (Princeton, New Jersey: Princeton Univ. Press, 1963).
- [18] L. Narici, E. Beckenstein. "Topological vector spaces" (New York: Marcel-Dekker Inc., 1985).
- [19] L.S. Pontrjagin. "Continuous groups" (Moscow: Nauka, 1984).
- [20] J.M. Souriau. "Groupes différentiels" (Berlin: Springer Verlag, 1981).
- [21] R. Sulanke, P. Wintgen. "Differentialgeometrie und Faserbündel" (Berlin: Veb deutscher Verlag der Wissenschaften, 1972).
- [22] W.H. Schikhof. "Ultrametric calculus" (Cambridge: Cambr. Univ. Press, 1984).
- [23] W.H. Schikhof. "Non-Archimedean calculus". Nijmegen: Math. Inst., Cath. Univ., Report **7812**, 130 pages, 1978.
- [24] V.S. Vladimirov, I.V. Volovich, E.I. Zelenov. " p -adic analysis and mathematical physics" (Moscow: Nauka, 1994).

Ludkovsky S.V. Department of Applied Mathematics MIREA, av. Vernadsky 78, Moscow 119454
 sludkowski@mail.ru